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A NOTE ON THE DUBINS-SAVAGE UTILITY OF A STRATEGY

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ABSTRACT

In their book, *How To Gamble, If You Must*, Lester E. Dubins and Leonard J. Savage defined the utility $\mu(\sigma)$ of a strategy σ with respect to a utility function u by $\mu(\sigma) = \limsup_{t\to\infty} \mu(\sigma, t) = \limsup_{t\to\infty} E(\sigma, u_t)$ where "lim sup" is taken over the directed set of all stop rules on H, u_t is the real-valued function on H defined by $u_t(h) = u(f_{t(h)})$ if $h = (f_1, f_2, \cdots f_{t(h)} \cdots)$ and $E(\sigma, u_t)$ is the Eudoxus integral of u_t with respect to the strategy σ. In this note, we will show that $\mu(\sigma)$ is nothing but the σ-integral of the real-valued function u^* defined on H by $u^*(h) = \limsup_{n\to\infty} u(f_n)$ if $h = (f_1, f_2, \cdots)$ in H. Finally, an application of this result is stated.

1. Introduction

Suppose that F is a non-empty set, $H = F^\infty = F \times F \times \cdots$, σ is a strategy on H, and u is a bounded, real-valued function defined on F. In [1], Dubins and Savage called the function u a utility function, and they defined the utility of the strategy σ with respect to the utility function u by, (c.f. pp. 24–25, 38–39 of [1]),

$$\mu(\sigma) = \limsup_{t\to\infty} \mu(\sigma, t) = \limsup_{t\to\infty} E(\sigma, u_t)$$

where $u_t(h) = u(f_{t(h)})$ if $H = (f_1, f_2, \cdots f_{t(h)} \cdots) \in H$, and "lim sup" is taken over the directed set of all stop rules defined on H.

In [4], Sudderth proved that, under the measurability assumptions on u and σ,

$$(1) \qquad \mu(\sigma) = \int u^*(h) d\sigma(h)$$

where $u^*(h) = \limsup_{n\to\infty} u(f_n)$ if $h = (f_1, f_2, \cdots f_n \cdots) \in H$. The main purpose

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of this paper is to prove that (1) still holds without the measurability assumptions for the strategy σ and the utility function u. An application of this result is also stated.

2. Main results

We state and prove a useful lemma before proving (1).

LEMMA 1. *Suppose that $A_1, A_2, \cdots A_m$ are m pairwise disjoint subsets of F and $a_1 > a_2 > \cdots > a_m > 0 (1 \leqq m < \infty)$. Suppose that u is a bounded, real-valued function defined on F by*

$$(*) \qquad u(f) = \sum_{i=1}^{m} a_i \chi_{A_i}(f).$$

Then $\mu(\sigma) = \Sigma_{i=1}^{m} a_i \sigma(B_i - C_{i-1})$, where $B_i = [h \mid h = (f_1, f_2, \cdots),\ f_k \in A_i$ for infinitely many $k]$, $C_0 = \varnothing$, $C_i = \cup_{j=1}^{i} B_j$, $i = 1, 2, \cdots m$.

PROOF. The proof of Lemma 1 is in three steps.

Step 1. For any stopping rule τ on H and each $i(=1,2,\cdots m)$, define

$$E_i^\tau = [h \mid h = (f_1, f_2, \cdots), \text{ for some } k \geqq \tau(h),\ f_k \in A_i]$$

$$F_0^\tau = \varnothing,\ F_i = \bigcup_{j=1}^{i} E_j^\tau, \text{ then}$$

$$\sup_{t \geqq \tau} u(\sigma, \tau) \leqq \sum_{i=1}^{m} a_i \sigma(E_i^\tau - F_{i-1}^\tau).$$

Here the "sup" is taken over all stop rules which are greater than or equal to τ.

To see this, let $t \geqq \tau$,

$$\mu(\sigma, t) = \int u(f_{t(h)}) d\sigma(h) = \int_{\cup_{j=1}^{m} E_j^\tau} u(f_{t(h)}) d\sigma(h)$$
$$= \sum_{i=1}^{m} \int_{E_i^\tau - F_{i-1}^\tau} u(f_{t(h)}) d\sigma(h) \leqq \sum_{i=1}^{m} a_i \sigma(E_i^\tau - F_{i-1}^\tau).$$

Step 2. Let $B_i - C_{i-1} = D_i$, $i = 1, 2, \cdots m$, then

$$\mu(\sigma) \leqq \sum_{i=1}^{m} a_i \sigma(D_i).$$

To see this, for each n ($=1,2,\cdots$), each i ($=1,2,\cdots m$), define a clopen set

$$K_i^{(n)} = [h \mid h = (f_1, f_2, \cdots),\ f_n \in A_i]$$

then $B_i = \cap_{m \geqq 1}^{\infty} \cup_{n \geqq m}^{\infty} K_i^{(n)}$.

For each stop rule s and each i ($i = 1,2,\cdots m$) define two open sets

$$A_i^{(s)} = [h \mid h \in \bigcup_{n \geqq s(h)} K_i^{(n)}], \quad D_i^{(s)} = \bigcup_{j=1}^{i} A_j^{(s)}.$$

Now, by Theorem 7–2 of [3], we have

$$\lim_{\tau \to \infty} \sigma(A_i^{(\tau)} - D_{i-1}^{(\tau)}) = \sigma(B_i - C_{i-1}).$$

By Step 1 and this result, we have

$$\mu(\sigma) \leqq \sum_{i=1}^{m} a_i \sigma(D_i).$$

Step 3. For any stop rule τ on H,

$$\sup_{t \geqq \tau} \mu(\sigma, t) \geqq \sum_{i=1}^{m} a_i \sigma(D_i).$$

To see this, let $\varepsilon > 0$, and inductively define the incomplete stop rules $t_1, t_2, \cdots t_m$ and stop rules $\hat{t}_1, \hat{t}_2, \cdots \hat{t}_m$ as follows.

For each i($=1,2,\cdots m$), let

$$t_i(h) = \text{the first } k \geqq \hat{t}_{i-1}(h) \text{ and } f_k \in A_i$$
$$= \infty \text{ if no such } k \text{ exists, where } h = (f_1, f_2, \cdots f_k \cdots).$$

By Corollary 3–2 of [3], for each i($=1,2,\cdots m$), we can and do choose a stop rule s_i on H such that $\sigma[t_i < \infty] \leqq \sigma[t_i \leqq s_i] + \varepsilon / a$ where $a = \Sigma_{i=1}^{m} a_i$. Let $\hat{t}_i = \max(s_i, \hat{t}_{i-1} + 1)$, therefore, we have

(i) $\tau = \hat{t}_0 < \hat{t}_1 < \cdots < \hat{t}_m$

(ii) $\sigma[t_i < \infty] \leqq \sigma[t_i \leqq \hat{t}_i] + \varepsilon / a$

(iii) $B_i - C_{i-1} = D_i \subseteq B_i \subseteq [t_i < \infty]$

(iv) $\sigma(D_i \cap [t_i \leqq \hat{t}_i]) \geqq \sigma(D_i) - \varepsilon / a$.

Now, let $t = t_1 \wedge t_2 \wedge \cdots \wedge t_m \wedge \hat{t}_m$, then $t \geqq \hat{t}_0 = \tau$, and if $t_i \leqq \hat{t}_i$ for some $i(1 \leqq i \leqq m)$, then $t = t_j$ for some $1 \leqq j \leqq i$. Hence

$$u(f_{t(h)}) = a_j \geqq a_i$$

$$\mu(\sigma, t) = \int u(f_{t(h)}) d\sigma(h) \geqq \sum_{i=1}^{m} \int_{D_i} u(f_{t(h)}) d\sigma(h)$$

$$\geqq \sum_{i=1}^{m} \int_{D_i \cap [t_i \leqq \hat{t}_i]} u(f_{t(h)}) d\sigma(h) \geqq \sum_{i=1}^{m} a_i \sigma(D_i \cap [t_i \leqq \hat{t}_i])$$

$$\geqq \sum_{i=1}^{m} a_i \left(\sigma[D_i] - \frac{\varepsilon}{a} \right) = \sum_{i=1}^{m} a_i \sigma(D_i) - \varepsilon.$$

Since ε is arbitrary, $\sup_{t \geqq} \mu(\sigma, t) \geqq \sum_{i=1}^{m} a_i \sigma(D_i)$. The proof of Lemma 1, now, is complete.

THEOREM 1. *Suppose that σ is a strategy on H, u is a utility function on F, then*

$$\mu(\sigma) = \int u^*(h) d\sigma(h)$$

where $u^*(h) = \limsup_{n \to \infty} u(f_n), h = (f_1, f_2, \cdots f_n \cdots) \in H$.

PROOF. Without loss of generality, we can and do assume u is non-negative. Choose a sequence of simple functions $\{u_k\}$ that converges uniformly to u from below. By Lemma 1,

$$\mu_n(\sigma) = \int u_n^*(h) d\sigma(h) \quad n = 1, 2, \cdots.$$

By Dominated Convergence Theorem, (pp. 124 of [2]),

$$\lim_{n \to \infty} \int u_n^*(h) d\sigma(h) = \int \lim_{n \to \infty} u_n^*(h) d\sigma(h)$$

$$= \int u^*(h) d\sigma(h).$$

For any $\varepsilon > 0$, there is an $N \geqq 1$ such that for all $n \geqq N$, $u_n(f_{t(h)}) \geqq u(f_{t(h)}) - \varepsilon$ for all h in H, and all stop rules t. Hence, if $n \geqq N$,

$$\mu_n(\sigma, t) = \int u_n(f_{t(h)})d\sigma(h) \geqq \mu(\sigma, t) - \varepsilon$$

$$\limsup_{t \to \infty} \mu_n(\sigma, t) \geqq \limsup_{t \to \infty} \mu(\sigma, t) - \varepsilon$$

$$\text{i.e., } \lim_{n \to \infty} \mu_n(\sigma) \geqq \mu(\sigma) - \varepsilon.$$

Since ε is arbitrary, $\lim_{n\to\infty}\mu_n(\sigma) \geqq \mu(\sigma)$.

Since $u_n \leqq u$ for all $n = 1, 2, \cdot\cdot 3 \cdot$, it is obvious $\mu_n(\sigma, t) \leqq \mu(\sigma, t)$ for all $n = 1, 2, \cdots$ and all stop rules t. Hence $\lim_{n\to\infty}\mu_n(\sigma) \leqq \mu(\sigma)$. Therefore, $\lim_{n\to\infty}\mu_n(\sigma) = \mu(\sigma) = \int u^*(h)d\sigma(h)$. The proof of Theorem 1, now, is complete.

3. An application

Now, we will give a simple proof of Theorem 3–2–1 of [1] by using Theorem 1.

Suppose that σ is a strategy on H, u is a utility function on F. Then

$$(*) \qquad \mu(\sigma) = \int_X \mu(\sigma[x])d\sigma_0(x).$$

Proof of (*)

$$\mu(\sigma) = \int_H u^*(h)d\sigma(h) \quad \text{(by Theorem 1)},$$

$$= \int_X \int_H (u^*x)(h')d\sigma[x](h')d\sigma_0(x) \quad \text{(By Fubini's Theorem)},$$

$$= \int_X \mu(\sigma[x])d\sigma_0(x),$$

since $(u^*x)(h') = u^*(xh') = u^*(h')$.
Hence $\mu(\sigma) = \int_X \mu(\sigma[x])d\sigma_0(x)$.

REMARK. Although Fubini Theorem in this usual form is not valid in all finitely additive settings, it is valid for bounded Borel measurable functions. We omit the justification here.

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